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LETTER TO THE EDITOR

A complete set of integrals in non-relativistic mechanics

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Abstract. The existence, number and type of the constants of motion in non-relativistic mechanics are examined for different set-ups of Newton's equations in configuration space, the Noetherian symmetries in the Lagrangian formulation, the Hamiltonian formulation and the Schrödinger equation in quantum theory and are found to be equivalent and exhaustive, as already known. Time-dependent constants are shown to be arbitrary, but nevertheless amenable to the general symmetry methods of Katzin, Levine and Mariwalla.

Recently† there have been a number of papers investigating various types of 'new' constants of motion in non-relativistic mechanics. There has also been a proliferation of papers on time-dependent constants of motion.

A few years ago, Katzin and Levine (1973, 1977) and, independently, Mariwalla (1975a, b, 1978, 1979a, b) gave a general procedure for investigating this problem; Mariwalla also gave a complete classification for a class of forces, and established a relation between energy conservation and dilation symmetry. The various constants discussed in the literature, therefore, could not be any different from the KLM constants. It is possible that this work has either not been seen, or has been ignored or misunderstood, or there may be other types of constants of motion not covered by this formalism. I give two alternate formulations of the problem which may appeal to readers more familiar with these formalisms, and show that the KLM constants *essentially* exhaust the list.

Consider an equation of Newton's type: $\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}; t)$. Envisage an infinitesimal transformation in configuration space of the type $\boldsymbol{x} \to \boldsymbol{x} + \epsilon\boldsymbol{\xi}$, $\mathrm{d}t \to \mathrm{d}t + \epsilon\dot{\xi}_0$ ($\dot{\xi}_0$ is in general not equal to $\mathrm{d}\xi_0/\mathrm{d}t$, but otherwise a dot will denote differentiation with respect to t, and $\boldsymbol{\partial} = \partial/\partial\boldsymbol{x}$, $\dot{\boldsymbol{\partial}} = \partial/\partial\dot{\boldsymbol{x}}$ leaving it unchanged:

$$\dot{\boldsymbol{x}} \cdot \delta(\ddot{\boldsymbol{x}} - \boldsymbol{F}) = \epsilon[\dot{\Lambda} - \tfrac{1}{2}\boldsymbol{\xi} \times \dot{\boldsymbol{x}} \cdot \nabla \times \boldsymbol{F} + (\Delta^j \ddot{x}^k - \dot{x}^j \dot{\Delta}^k)\dot{\partial}_k F_j + \Delta^j \partial_t F_j] = 0$$

$$\Lambda \equiv \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{\Delta}} - \ddot{\boldsymbol{x}} \cdot \boldsymbol{\Delta} \equiv \dot{\boldsymbol{\xi}} \cdot \dot{\boldsymbol{x}} - \ddot{\boldsymbol{x}} \cdot \boldsymbol{\xi} - 2\dot{\boldsymbol{\xi}}_0 \dot{\boldsymbol{x}}^2 \qquad \boldsymbol{\Delta} = \dot{\boldsymbol{\xi}} - \xi_0 \dot{\boldsymbol{x}}.$$

For $\boldsymbol{F} = -\nabla\phi(\boldsymbol{x})$ all but the $\dot{\Lambda}$ term vanish identically. Two cases arise. In cases where the symmetry admitted is an isometry of the background space ($\boldsymbol{\xi} = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b}$, $\xi_0 = a_0$), the symmetry is a constraint on the system so that the relevant Λ vanishes identically and by d'Alemberts' principle of virtual work $\delta\boldsymbol{x} \cdot (\ddot{\boldsymbol{x}} - \boldsymbol{F}) = 0$; on integration, this would give the conservation laws of energy and momenta. When the infinitesimal symmetries are *not* isometries of background space and $\boldsymbol{F} = -\nabla\phi(\boldsymbol{x})$ quadratic conservation laws given by Λ result. *These are the KLM constants.* (If $\boldsymbol{F} \neq -\nabla\phi$, then in suitable cases

† For representative literature on the subject, see the recent papers by Gonzalez-Gascon (1980), Leach (1980) and Dekkar (1980).

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constants of motion $\Lambda + f(x, t)$ would still exist for some f.) The traceless tensor for the potential x^2, the vector for the inverse r potential and the energy for the potential r^α are examples of KLM constants (Mariwalla 1973, 1975a, b, 1978, 1979a, Katzin 1973, 1977).

Let $\tilde{L} = JL$ be a Lagrange density of the system and $\boldsymbol{I}$ the Euler–Lagrange operator; then

$$\delta\tilde{L} = J(\boldsymbol{\Delta} \cdot \boldsymbol{I}L + \dot{\Omega}) \qquad \Omega = L\xi_0 + \boldsymbol{\Delta} \cdot \dot{\partial} L$$

and if Euler–Lagrange equations are satisfied

$$\dot{q} \cdot \delta(\boldsymbol{I}L) = \mathrm{d}\delta H/\mathrm{d}t + \partial\delta\tilde{L}/\partial t \qquad \delta H = (\dot{\boldsymbol{q}} \cdot \dot{\partial} - 1)\,\delta L = \epsilon\Lambda.$$

For isometries, again $\Lambda \equiv 0$ and $\delta\tilde{L}$ vanishes identically; in fact, if L has no explicit time dependence and is spherically symmetric then

$$\boldsymbol{\Delta} \cdot \boldsymbol{I}L = -\dot{\Omega} - a_0 \partial L/\partial t + \boldsymbol{a} \cdot (\dot{\boldsymbol{q}} \times \dot{\boldsymbol{\partial}} - \boldsymbol{q} \times \boldsymbol{\partial})L \qquad \Omega = \boldsymbol{a} \cdot \boldsymbol{L} + a_0 H + \boldsymbol{b} \cdot p.$$

Thus momenta $\boldsymbol{p} = \dot{\partial}L$, $\boldsymbol{L} = \boldsymbol{q} \times \boldsymbol{p}$ and energy $H = \dot{\boldsymbol{q}} \cdot \boldsymbol{p} - L$ are conserved if the Euler–Lagrange equations are satisfied and L is unchanged under the relevant (space translation, rotation, time translation) symmetry. The Λ's are the KLM constants. Both Ω (*d'Alembert's constants*) and Λ are clearly Noetherian symmetries (apart from an 'arbitrary' additive function f such that $\delta\tilde{L} = \epsilon\, \mathrm{d}f/\mathrm{d}t$ (Mariwalla 1978, 1979a)).

The transformations in (Lagrangian/Hamiltonian) phase space are related to those of the configuration space (and its prolongations) considered above by the relevant conserved object, say $K(=\Omega, \Lambda)$ being the generator of the infinitesimal (canonical) transformation. Thus in phase space (Mariwalla 1979a)

$$\boldsymbol{q} \to \boldsymbol{q} + \epsilon\boldsymbol{\mu} \qquad \dot{\boldsymbol{q}} \to \dot{\boldsymbol{q}} + \epsilon\boldsymbol{\nu} \qquad \boldsymbol{\mu} = -\dot{\boldsymbol{\partial}}K \qquad \boldsymbol{\nu} = \boldsymbol{\partial}K$$

(where, for the Hamiltonian formulation, $\dot{\boldsymbol{q}}$ is replaced by $\boldsymbol{p}$).

The results also hold for a curved background space. If $\boldsymbol{\nabla}$ is the covariant derivative, Newton's equation reads $\mathrm{D}\dot{\boldsymbol{q}} = (\dot{\boldsymbol{q}} \cdot \boldsymbol{\nabla})\dot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{x})$. Its Lie derivative $£(\mathrm{D}\dot{\boldsymbol{q}} - \boldsymbol{F}) \equiv \boldsymbol{h}$ (with respect to $\boldsymbol{\xi}$, ξ_0) when put to zero yields as before the formula

$$\dot{\Lambda} = \tfrac{1}{2}\dot{\boldsymbol{q}} \times \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \times \boldsymbol{F}$$

in covariant language (Mariwalla 1975a, b, 1979a).

All these considerations merely tell us that, if integrals of motion exist, what their possible forms are and the relation between the corresponding infinitesimal transformations in configuration and phase spaces. They do *not* tell us how to find them (principle of Relativity of Paths). The KLM method gives a unique (up to equivalence) prescription for doing this. Find the automorphisms admitted by the background space characterised by the invariance of paths. Let S be an intrinsic geometric parameter along these paths and $\alpha = \mathrm{d}t/\mathrm{d}s$. Then the system admits a vector field $(\boldsymbol{\xi}, \xi_0)$ belonging to the automorphism group of background space, provided

$$£\boldsymbol{F} + 2\dot{\xi}_0\boldsymbol{F} + (\ddot{\xi}_0 - \dot{\psi})\dot{\boldsymbol{q}} + k\dot{\boldsymbol{q}}^2\boldsymbol{\nabla}\psi = 0, \tag{A}$$

where $k = +1$ for conformal motions and zero for (projective) collineations. The method naturally holds also for relativisitic systems where conformal symmetries would arise for the paths of massless particles (Mariwalla 1975b).

One may also express these results in the language of vector fields in a natural base. Let $F = \boldsymbol{F} \cdot \boldsymbol{\partial}$, $\xi = \boldsymbol{\xi} \cdot \boldsymbol{\partial}$, $\mathscr{F} = (£\boldsymbol{F}) \cdot \boldsymbol{\partial}$ where $\boldsymbol{F} = \boldsymbol{F}(x)$; then the system admits a symmetry $\boldsymbol{\xi}$ if and only if the commutator $[\xi, F] = \mathscr{F} = 0$. For transformations involving t

and for curved spaces or generalised coordinates, we obtain in the prolonged space (Mariwalla 1979a)

$$[h, \xi] = \dot{\xi}_0 h + \boldsymbol{h} \cdot \dot{\boldsymbol{\partial}} \qquad h = \dot{\boldsymbol{q}} \cdot \boldsymbol{\partial} + \ddot{\boldsymbol{q}} \cdot \dot{\boldsymbol{\partial}} + \partial_t \qquad \xi = \boldsymbol{\xi} \cdot \boldsymbol{\partial} + \boldsymbol{\eta} \cdot \dot{\boldsymbol{\partial}}$$

where

$$\epsilon \boldsymbol{\eta} = \delta \dot{\boldsymbol{q}} \qquad \boldsymbol{h} = \pounds(\mathrm{D}\dot{\boldsymbol{q}} - \boldsymbol{F}).$$

The vanishing of $\boldsymbol{h}$ together with the condition (A) gives the necessary and sufficient conditions for the existence of symmetries and related constants of motion. In (Lagrangian) phase space, if we put $\lambda = \boldsymbol{\mu} \cdot \boldsymbol{\partial} + \boldsymbol{\nu} \cdot \dot{\boldsymbol{\partial}}$, then the conditions are $[h, \lambda] = 0$, $h\Lambda = 0 = \lambda E$. In Hamiltonian phase space (replacing $\dot{\boldsymbol{q}}$ by $\boldsymbol{p}$) $h \equiv \pounds_H$, $\lambda = \pounds_\Lambda$ and the conditions are

$$[h, \lambda] = \pounds_\sigma = 0 \qquad \sigma = \{H, \Lambda\}_{\mathrm{PB}} \qquad h\Lambda = 0 = \lambda H.$$

The additional generators leading to a non-invariance group are given by $\Sigma_\Lambda \equiv \pounds_\Lambda(\boldsymbol{p} \cdot \dot{\boldsymbol{q}}) = \mathrm{d}(\Lambda + \boldsymbol{\mu} \cdot \boldsymbol{p})/\mathrm{d}t$ and are constants of motion for a system with a potential which is the negative of the one for which Λ is a constant of motion. The non-invariance groups $\mathrm{Sp}(n;\boldsymbol{R})$ and SU(4;2) for potentials $\boldsymbol{x}^2$ and r^{-1} arise in this way (Mariwalla 1975a, b, 1978, 1979a).

Theorem. (i) The maximal number of independent conserved objects cannot be greater than $n^2(n^2+n$ in an affine scheme). (ii) The dimension of a non-invariance group is at most that of $\mathrm{Sp}(n;\boldsymbol{R})$ ($2n^2+3n$ in an affine scheme). The first of these follows from the fact that at a point one can prescribe at most n linearly independent vector fields. Thus for a free particle the symmetry group is $\mathrm{SL}(n+1;\boldsymbol{R})$ of dimension n^2+2n; the equation of motion involves $2n$ arbitrary constants leaving n^2 constants of motion at most. The stability subgroup of the symmetry group of Newton's equation is the same as that for symmetries in phase space (for these the KLM constant $\Lambda \equiv 0$). For spherical symmetry, which is maximal in some sense, we obtain $3n^2 - 2(n^2-n)/2 = 2n^2+n =$ dimension of $\mathrm{Sp}(n;\boldsymbol{R})$. We note that the affine symplectic group on the reals is the full canonical group leaving the symplectic structure unchanged and gives the non-invariance group in an affine scheme (Mariwalla 1979a).

The history† of the *time-dependent* constants of motion goes back to the classical problem of 'compatibility' between certain systems of differential equations; as we shall see these are not in general 'genuine' constants of motion. For instance, for the equation $\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}; t)$, if f_1, f_2 are its two linearly independent solutions (or of a 'related equation') and $\boldsymbol{x}(t)$ its general solution, then there would exist several functions $M(\boldsymbol{x}, \dot{\boldsymbol{x}}, f_1, f_2, \dot{f}_1, \dot{f}_1 \ldots; t)$ such that $\dot{M} = 0$. The point at issue is that $\boldsymbol{x}, \dot{\boldsymbol{x}}$ are in the final analysis to be determined as functions of time t, so that a concommitant of $\boldsymbol{x}, \dot{\boldsymbol{x}}$, say a polynomial together with a suitable choice of time-dependent coefficients, can always be made time independent. All the examples discussed in the literature belong to one of these cases. Thus, for a harmonic oscillator, we get constants

$$\boldsymbol{x} \cos \omega t - \dot{x} \sin \omega t \qquad a\boldsymbol{x}^2 + b\dot{\boldsymbol{x}}^2 + c\boldsymbol{x} \cdot \dot{\boldsymbol{x}}$$

where $c = \dot{a} = -\dot{b}$ and $\ddot{a} + 4a = \text{constant}$. One can also take a, b, c as quadratic functions of solutions of $\ddot{f} + 4f = 0$ to obtain another quadratic integral, etc. For the

† One may trace it as far back as Elie Cartan's first (?) work in 1899 (Cartan 1950–5). See also Thomas (1952).

time-dependent case $\omega = \omega(t)$, we get by the same procedure the constants

$$f\dot{x} - \dot{f}x \qquad \tfrac{1}{2}(\ddot{b} + 2b\omega^2)x^2 + b\dot{x}^2 - \dot{b}x \cdot \dot{x}$$

where $\ddot{f} + \omega^2 f = 0$ and $\dddot{b} + 4\dot{b}\omega^2 + 4b\dot{\omega}\omega = 0$, etc. The Lewis (1968) constant also arises in the same manner. In any case, there is no obvious limit to the number of such constants, and it would clearly depend on the number of the other systems 'compatible' with the given one. Nevertheless it is possible to bring these, in a measure, within the ambit of the KLM scheme by considering the configuration space to be $(n+1)$-dimensional space–time. This is always possible, as a free-particle Lagrangian under the change $\boldsymbol{x} \to \boldsymbol{u}(\boldsymbol{x}, t)$ takes the form

$$L = g_{ij}(\boldsymbol{x}, t)\dot{x}^i\dot{x}^j + h_j(\boldsymbol{x}, t)\dot{x}^j + f(\boldsymbol{x}, t).$$

The projectivities now are in the $(n+1)$ dimensions, and for flat space give $SL(n+2; \boldsymbol{R})$; accordingly, the independent constants of motion would be $\leqslant (n+1)^2$. In fact, the various theorems above will hold with the formal change $n \to n+1$. But from a physical viewpoint, as seen above, these constants would mostly be irrelevant. Their arbitrariness, however, is ingeniously related to the existence of gauge-inequivalent Lagrangians for a given equation of motion. The vector fields $(\boldsymbol{\xi}, \dot{\xi}_0)$ generating the symmetries will correspondingly differ from those of the time-independent case by having time-dependent coefficients in place of time-independent ones (this is suggested by the Lagrangian above). Thus the Lewis constant would arise from (time-dependent) dilation symmetry and is an anologue of the energy integral. The other constants are similarly interpreted together with their invariance and non-invariance groups in $(2n+2)$-dimensional phase space.

Locally, these considerations extend to velocity-dependent equations of the type

$$\ddot{\boldsymbol{x}} + \boldsymbol{b}_{jk}\dot{x}^j\dot{x}^k + c\dot{\boldsymbol{x}} + \nabla\phi = 0.$$

By a coordinate change one can eliminate the quadratic terms locally on a section of constant Riemannian curvature: positive, negative or zero. The linear term is similarly eliminated locally by a change in the time parameter. The resulting equation has the form $\boldsymbol{q}'' + c(T)\boldsymbol{\partial}\chi = 0$, and can be handled as above and its quantum theory set up unambigously. In this form one could also give a satisfactory local treatment of a 'coherent state representation' of the problem (Mariwalla 1979b).

It is amusing that these considerations on symmetries and conservation laws extend, *mutatis mutandis*, to quantum theory (Mariwalla 1975b).

Theorem. Let $M_A = \{\psi_{A_j}\}$ denote the manifold of solution with the same energy eigenvalue E_A of the operator $\mathcal{O} = \boldsymbol{H} - \boldsymbol{E}$. Let $U_\alpha = \exp(i\alpha\boldsymbol{\omega})$ be a unitary transformation such that $\psi_{A_1}(\boldsymbol{x}, t) \to \psi_{A_2}(x', t')\exp(if(x', t'))$; then we get two possiblities.

(i) $\mathcal{O}\psi = 0$ admits a symmetry u_α if and only if

$$\mathcal{O}(\boldsymbol{\omega}\psi) = 0 \qquad [\boldsymbol{\omega}, \mathcal{O}]\psi = f(\boldsymbol{x}, t)(\mathcal{O}\psi).$$

(ii) Let

$$\pounds_\omega \mathcal{O} = \lim_{x' \to 0} (u_\alpha \mathcal{O} u_\alpha^{-1} - \mathcal{O})/\alpha = \boldsymbol{\Lambda};$$

then $\mathcal{O}\psi = 0$ admits a symmetry *induced by* u_α if and only if

$$[\mathcal{O}, \boldsymbol{\Lambda})\psi = 0 \qquad \mathcal{O}\boldsymbol{\Lambda}\psi_{A_1} = \mathcal{O}\psi_{A_2}.$$

The proof consists in expanding $u_\alpha \mathcal{O}\psi$ and rearrangement of terms of first order in α, etc. The space–time isometries of course leave $\mathcal{O}$ unchanged, so that $\Lambda \equiv 0$ in this case.

The pure Galilean transformations, linear fractional transformations of time and the Trautman maps all belong to case (i) and reflect the fact that the Schrödinger equation is a representation of the Galilean group in the projective model of Euclidean 'velocity space'. The case (ii) describes the connection between configuration space symmetries, symmetries of the Schrödinger operator and the degeneracy of its eigenvalues. The set of all (independent) operators Λ obtained in the above manner and which commute with the Hamiltonian would connect different degenerate states with each other. The space–time transformation $u_\alpha = \exp(\mathrm{i}\alpha\boldsymbol{\omega})$ on the other hand, is a symmetry of the corresponding Newtonian equation of motion. The unique importance of this result need hardly be stressed (Mariwalla 1975b, 1978, 1979a).

The content of the relation between symmetries and conservation laws is found to be identical in the formulation of Newtonian configuration space, Lagrangian–Noetherian, canonical Hamiltonian and quantum-mechanical Schrödinger equations. The method of Katzin, Levine and Mariwalla, which also gives this unified picture, permits the deduction of all symmetries from geometric considerations, and clarifies the much misunderstood relation between symmetries in configuration and phase space, including the connection between the non-invariance group in phase space and invariance of the free Schrödinger and Newtonian equations under linear fractional trnasformations. The methods extend to relativity and to field theory (Mariwalla 1975b, 1978, 1979a).

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